Q-G Potential Vorticity

$$q = \nabla_h^2 \psi + f + \frac{1}{\rho_0} \frac{\partial}{\partial z^*} \left(\varepsilon \rho_0 \frac{\partial \psi}{\partial z^*} \right)$$

$$\varepsilon = f_0^2 / N^2$$

$$u = -\frac{\partial \psi}{\partial y}$$

$$v = +\frac{\partial \psi}{\partial x}$$

Section 12.3.1

$$\left(\frac{\partial}{\partial t} + \vec{V}_g \cdot \nabla\right) q = 0$$

$$q \equiv \nabla^2 \psi + f(y) + \frac{f_0^2}{N^2} \frac{1}{\rho_0} \frac{\partial}{\partial z^*} \left(\rho_0 \frac{\partial \psi}{\partial z^*} \right)$$

$$\psi = \Phi / f_0$$

$$q = \bar{q} + q'$$

$$\psi = -\bar{u}y + \psi'$$

Basic state:
constant zonal mean
flow

$$\bar{q} = \cancelto{0}{\nabla^2 \bar{\psi}} + f_0 + \beta y + \frac{f_0^2}{N^2}\,\cancelto{0}{\frac{1}{\rho_0}\frac{\partial}{\partial z^*}\left(\rho_0 \frac{\partial \bar{\psi}}{\partial z^*}\right)}$$

$$\frac{\partial}{\partial y}\overline{q} = \frac{\partial}{\partial y}f$$

$$= \beta$$

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)q' + \beta\frac{\partial\psi'}{\partial x} = 0$$

$$q' = \nabla^2\psi' + \frac{f_0^2}{\rho_0 N^2}\frac{\partial}{\partial z^*}\left(\rho_0\frac{\partial\psi'}{\partial z^*}\right)$$

solutions for waves have form:

$$\psi'(x, y, z^*, t) = \Phi(z^*) e^{i(kx + ly - kc_x t) + \frac{z^*}{2H}}$$

Note:

$$\frac{\partial \psi'}{\partial z^*} = \frac{\partial \Psi}{\partial z^*}\left(\frac{\psi'}{\Psi}\right) + \frac{\psi'}{2H}$$

$$\frac{\partial^2 \psi'}{\partial z^{*2}} = \frac{\partial^2 \Psi}{\partial z^{*2}}\left(\frac{\psi'}{\Psi}\right) + \frac{\partial \Psi}{\partial z^*}\left(\frac{\psi'}{2H}\right) + \frac{1}{2H}\frac{\partial \Psi}{\partial z^*}\frac{\psi'}{\Psi} + \frac{1}{4H^2}\psi'$$

$$q' = -(k^2+l^2)\psi' + \frac{f_0^2}{N^2}\psi'\left\{\frac{1}{\bar{\Psi}}\frac{\partial^2\bar{\Psi}}{\partial z^{*2}} - \frac{1}{4H_\rho^2}\right\}$$

$$\frac{\partial \Psi}{\partial z^{*2}} + m^2 \Psi = 0$$

$$m^2 = \frac{N^2}{f_0^2}\left[\frac{\beta}{(\bar{u} - c_x)} - (k^2 + l^2)\right] - 1/4H^2$$