Alternative Vertical Coordinates

z system: x, y, z, t
(all independent var.)
transform to alternate:
x, y, s, t where

$$s = s(x, y, z, t)$$

(1) $S = S(x, y, z, t)$

Require S monotonic with z

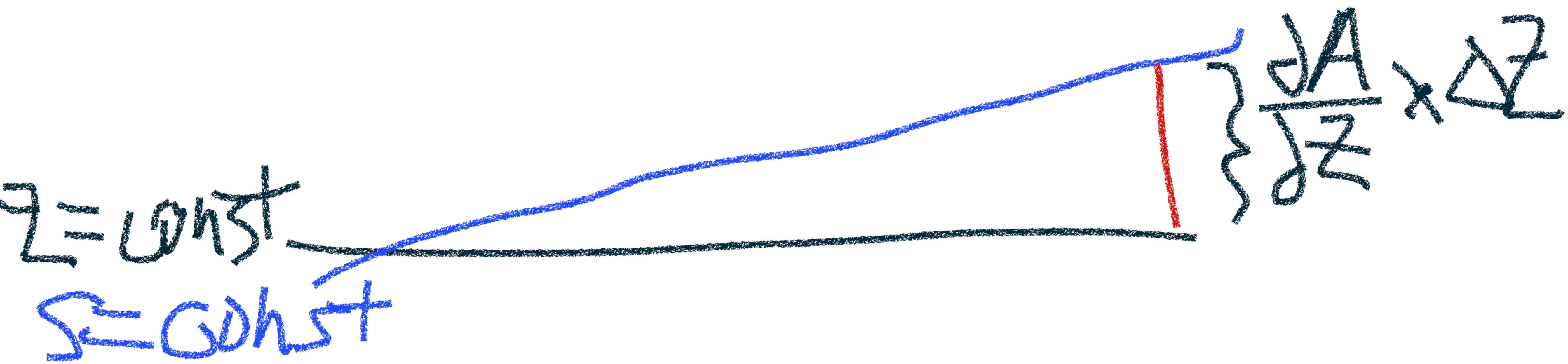
(1) $\Longrightarrow$

$$z = z(x, y, S, t)$$

Consider scalar A as a 4-d variable, using either z or s.

Then e.g.:

$$(2)\quad \left(\frac{\partial A}{\partial x}\right)_s = \left(\frac{\partial A}{\partial x}\right)_z + \frac{\partial A}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s$$



Consider

$$\frac{\partial A}{\partial z} = \frac{\partial A}{\partial S}\left(\frac{\partial S}{\partial z}\right)$$

Then (2) $\longrightarrow$

$$\left(\frac{\partial A}{\partial x}\right)_S = \left(\frac{\partial A}{\partial x}\right)_z + \frac{\partial S}{\partial z}\left(\frac{\partial z}{\partial x}\right)_S \left(\frac{\partial A}{\partial S}\right)$$

Similarly

$$\left(\frac{\partial A}{\partial t}\right)_s = \left(\frac{\partial A}{\partial t}\right)_z + \frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial t}\right)_s \frac{\partial A}{\partial s}$$

— or —

$$\left(\frac{\partial A}{\partial t}\right)_z = \left(\frac{\partial A}{\partial t}\right)_s - \frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial t}\right)_s \frac{\partial A}{\partial s}$$

- Note: now have "s" terms and "z" terms on opposite sides.

Then d/dt becomes

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbb{V}\cdot\nabla + \omega\frac{\partial}{\partial z}$$

$$= \left(\frac{\partial}{\partial t}\right)_s + \mathbb{V}\cdot\nabla_s + \left[\omega - \left(\frac{\partial z}{\partial t}\right)_s - \mathbb{V}\cdot\nabla_s z\right]\frac{\partial s}{\partial z}\frac{\partial}{\partial s}$$

But

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_s + \mathbf{V}\cdot\nabla_s + \dot{s}\frac{\partial}{\partial s}$$

$$\dot{s} = \frac{ds}{dt}$$

(like $w = dz/dt$)

So then we can relate the vertical speeds

$$\dot{s} = \frac{\partial s}{\partial z}\left[w - \left(\frac{\partial z}{\partial t}\right)_s - V\cdot\nabla_s z\right]$$

Horizontal momentum

$$\frac{\partial \mathbf{v}}{\partial t} + f\hat{k}\times\mathbf{v} = -\frac{1}{\rho}\nabla_s p + \frac{1}{\rho}\left(\frac{\partial s}{\partial z}\right)\nabla_s z\,\frac{\partial p}{\partial s} + \vec{F}$$

But

$$\frac{\partial p}{\partial z} = \left(\frac{\partial s}{\partial z}\right)\left(\frac{\partial p}{\partial s}\right)$$

$$-\rho g =$$

So, horizontal momentum becomes

$$\frac{d}{dt}\mathbb{V} + f\hat{k}\times\mathbb{V} = -\frac{1}{\rho}\nabla_s p - g\nabla_s z + \vec{F}$$

Pressure coordinate

$$\omega \equiv dp/dt$$

So then

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_p + \mathbb{V}\cdot\nabla_p + \omega\frac{\partial}{\partial p}$$

Then with s=p,

$$\frac{\partial \vec{V}}{\partial t} + f\hat{k}\times\vec{V} = \cancelto{0}{-\frac{1}{\rho}\nabla_s p}$$

$$+ \underbrace{\frac{1}{\rho}\left(\frac{\partial s}{\partial z}\right)\nabla_s z\,\frac{\partial p}{\partial s}}_{= -g\nabla_p z} + \vec{F}$$

Or

$$\frac{d\mathbb{V}}{dt} + f\hat{k} \times \mathbb{V} = -g\nabla_p Z + \vec{F}$$

Continuity Equation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbb{V}) + \frac{\partial}{\partial z}(\rho w) = 0$$

or

$$\frac{d}{dt}\ln \rho + \nabla \cdot \mathbb{V} + \frac{\partial w}{\partial z} = 0$$

Change to p-coord.
Key factor:

$$\frac{d}{dt}(\ln \rho) \longrightarrow$$

$$\frac{d}{dt} \ln\left(\rho \frac{\partial z}{\partial p}\right)$$

But $\partial z / \partial p = -1/\rho g$

So

$$\frac{d}{dt}\ln\left(\rho\frac{\partial z}{\partial p}\right)=\frac{d}{dt}\ln\left(-\rho/\rho g\right)$$

$$=0$$

Mass continuity is then

$$\nabla_p \mathbb{V} + \frac{\partial}{\partial p}\omega = 0$$

Other coordinates

Using p_T = constant p top of atmosphere, and p_H = surface pressure:

$$\sigma = (p - p_T)/p_*$$

$$p_* = p_H - p_T$$

$\sigma = 1$ @ p_H
$\sigma = 0$ @ p_T

Surface at $\sigma = 1$ everywhere