PROBLEMS 3 – CLIMATE MODELING

Due: 15 October 2015

1. Let’s see how well various spatial distributions are represented by Legendre polynomials. We want to construct representations for functions that are symmetric about the equator (x=0), so we can assume they are can be represented by the form

\[ f(x) \approx A_0 P_0(x) + A_2 P_2(x) + ... + A_n P_n(x) \]  \hspace{1cm} (1)

The relationship is exact if \( n \to \infty \).

We can compute the coefficients \( A_0, A_2, ..., A_n \) by recalling that

\[
\int_0^1 P_n(x) P_n'(x) dx = \delta_{n,n'} \left( \frac{1}{2n+1} \right) = \begin{cases} \frac{1}{2n+1} & n' = n \\ 0 & n' \neq n \end{cases} \]  \hspace{1cm} (2)

so then

\[
\int_0^1 f(x) P_n(x) dx = \int_0^1 \left( A_0 + A_2 P_2(x) + ... + A_n P_n(x) \right) P_n(x) dx = A_n \left( \frac{1}{2n+1} \right) \]  \hspace{1cm} (3)

yielding

\[
A_n = (2n+1) \int_0^1 f(x) P_n(x) dx \]  \hspace{1cm} (4)

For this problem, recognize that the first 4 even Legendre polynomials are

\[
P_0(x) = 1 \hspace{1cm} P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3) \]

\[
P_2(x) = \frac{3x^2 - 1}{2} \hspace{1cm} P_6(x) = \frac{1}{16} (231x^6 - 315x^4 + 105x^2 - 5) \]

For each case below, make a plot of the temperatures given by \( f(x) \) and then, on the same graph, a plot of

\[ F(x) = A_0 P_0(x) + A_2 P_2(x) + ... + A_n P_n(x) \]

using the \( A_n \) derived from the function. You should also give me to the values of the \( A_n \) needed for the plot.

(a) \( f_1(x) = 288 \text{ K} \) (i.e., the average temperature of the earth).

Compute coefficients \( A_n \) for \( n = 0, 2, 4, 6 \). You can do this one without resorting to computing equation (4) if you think carefully about equation (2) and how \( f_1 \) varies with \( x \).

(b) \( f_2(x) = 302 - 42x^2 \)

Compute coefficients for \( n = 0, 2, 4 \) to produce \( F(x) \)
(c) \( f_3(x) = \begin{cases} 
288 & 0 \leq x \leq 0.5 \\
268 & x > 0.5 
\end{cases} \)

Compute coefficients for \( n = 0, 2, 4, 6 \). Plot 3 curves: \( f_3(x) \), \( F(x) \) using just \( n=0 \) and 2, and then \( F(x) \) using \( n = 0, 2, 4 \) and 6. Does the second \( F \) always give a closer match to \( f_3 \) than the first?

(d) Why do we only need to use the even polynomials for the functions examined here? That is, why aren’t we also using \( P_n(x) \) for \( n = 1, 3, 5, \ldots \)?