Vorticity Equation Derivation

u-component $\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \right) \right]$

$$\underset{(1a)}{\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right)} + \underset{(2a)}{\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}} + \underset{(3a)}{u\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)} + \underset{(4a)}{\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}} + \underset{(5a)}{v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)} + \underset{(6a)}{\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}} + \underset{(7a)}{w\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)} - \underset{(8a)}{\frac{\partial f}{\partial y}v} - \underset{(9a)}{f\frac{\partial v}{\partial y}} = \underset{(10a)}{\frac{1}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}} - \underset{(11a)}{\frac{1}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)}$$

v-component $\left[\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \right) \right]$

$$\underset{(1b)}{\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right)} + \underset{(2b)}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}} + \underset{(3b)}{u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)} + \underset{(4b)}{\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}} + \underset{(5b)}{v\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)} + \underset{(6b)}{\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}} + \underset{(7b)}{w\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z}\right)} + \underset{(8b)}{\frac{\partial f}{\partial x}u} + \underset{(9b)}{\frac{\partial u}{\partial x}f} = \underset{(10b)}{\frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}} - \underset{(11b)}{\frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right)}$$

* subtract the u-component from the v-component
* Recall, $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \zeta$
* Also, remember that for partial derivatives we can switch order of differentials!!

→ taking (1b) − (1a) yields $\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial \zeta}{\partial t}$ (1)

→ taking (3b) − (3a) yields $u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) - u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = u\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = u\frac{\partial \zeta}{\partial x}$ (2)

→ taking (5b) − (5a) yields $v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) - v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = v\frac{\partial}{\partial y}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = v\frac{\partial \zeta}{\partial y}$ (3)

→ taking (7b) − (7a) yields $w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}\right) - w\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right) = w\frac{\partial}{\partial z}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = w\frac{\partial \zeta}{\partial z}$ (4)

→ taking (2b) − (2a) yields $\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = \frac{\partial u}{\partial x}(\zeta)$

→ taking (4b) − (4a) yields $\frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = \frac{\partial v}{\partial y}(\zeta)$

combine → $\zeta\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]$

→ taking (9b) − (9a) yields $f\frac{\partial u}{\partial x} - -f\frac{\partial v}{\partial y} = f\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]$

Combine → $(\zeta + f)\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]$ (5)

→ taking (6b) − (6a) yields $\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$ (6)

→ taking (8b) − (8a) yields $u\frac{\partial f}{\partial x}^{\to 0} - -v\frac{\partial f}{\partial y} = v\frac{\partial f}{\partial y}$ (7)

→ taking (11b) − (11a) yields $-\frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) - -\frac{1}{\rho}\frac{\partial}{\partial x}\frac{\partial p}{\partial y} = 0$

→ taking (10b) − (10a) yields $\frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{1}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x} = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$ (8)

→

* We are left with ----

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + (\zeta + f)\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{\partial f}{\partial y} = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

→ Recall, $\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} = \frac{d}{dt}(\zeta)$

→ Also, $\frac{df}{dt} = \cancelto{0}{\frac{\partial f}{\partial t}} + u\cancelto{0}{\frac{\partial f}{\partial x}} + v\frac{\partial f}{\partial y} + w\cancelto{0}{\frac{\partial f}{\partial z}}$

thus, $v\frac{\partial f}{\partial y} = \frac{d}{dt}(f)$

COMBINE THESE TERMS AND REWRITE EQUATION ABOVE

$$\underbrace{\frac{d}{dt}(\zeta + f)}_{\text{Rate of change in the absolute vorticity following the motion}} = \underbrace{-(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{\text{divergence term}} \underbrace{- \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)}_{\text{tilting or twisting term}} + \underbrace{\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)}_{\text{Solenoidal term}}$$