Derivation of the momentum equation in rotating coordinates

Newton’s 2nd law (i.e. the rate of change of momentum, measured relative to coordinates fixed in space, equals the sum of all forces) can be written symbolically as:

\[
\frac{d}{dt} \vec{V}_a = \sum F
\]  

(1)

which means that the rate of change of absolute velocity, \( \vec{V}_a \), following the motion in an inertial (fixed) coordinate system is equal to the sum of the real forces acting per unit mass.

Previously, we found through simple physical reasoning that when the motion is viewed in a rotating coordinate system certain additional apparent forces (coriolis and centrifugal) must be included in order to apply Newton’s 2nd law. We can obtain the same results through a formal transformation of the coordinates in Eq. (1).

To do this we must first find a relationship between \( \vec{V}_a \) and the velocity relative to the rotating system, which we can denote by \( \vec{V}_r \). To obtain this relationship, apply the equation we derived in the previous section,

\[
\frac{d}{dt} \vec{A} = \frac{d}{dt} \vec{\Omega} \times \vec{A},
\]  

(2)

to a position vector \( \vec{r} \) for an air parcel on the rotating Earth:

STEP 1:

Rewrite the expression in STEP 1 substituting \( \frac{d}{dt} \vec{r} = ? \) and \( \frac{d}{dt} \vec{A} = ? \)

(Hint: \( \frac{d}{dt} \vec{r} \) is similar to \( \frac{dx}{dt} \).)

STEP 2:

(1 point)
State, in words, what the expression derived in STEP 2 tells us:

(2 points)

If we apply Eq. (2) to the velocity vector \( \vec{V}_a \) we obtain:

\[
\frac{da\vec{V}_a}{dt} = \frac{d\vec{V}_a}{dt} + \vec{\Omega} \times \vec{V}_a
\]  

(3)

Substitute the expression from STEP 2 into Eq. (3) and derive the expression,

\[
\frac{da\vec{V}_a}{dt} = \frac{d\vec{V}}{dt} + 2\vec{\Omega} \times \vec{V} - \vec{\Omega} \times \vec{r},
\]  

(4)

assuming that \( \vec{r} \) represents a vector perpendicular to the axis of rotation, with magnitude equal to the distance to the axis of rotation.

[Hint: You will need to use the vector identity, \( \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \), applied to the expression, \( \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \)]

STEP 3:

(3 points)
Eq. (4) tells us that the acceleration following the motion in an inertial system (a) equals the rate of change of relative velocity following the relative motion in the rotating (b) frame plus the coriolis acceleration due to relative motion in the rotating frame (c) plus the centripetal acceleration caused by the rotation of the coordinates (d). **Rewrite Eq. (4) and identify terms (a), (b), (c), and (d).**

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(1 point)

If we assume that the only real forces acting on the atmosphere are the pressure gradient force, gravitational force, and frictional force, we can rewrite Eq. (1) as:

\[
\frac{d_a \mathbf{\tilde{V}}_a}{dt} = \sum F = -\frac{1}{\rho} \nabla p + g^s + F_r
\]  

(5)

By substituting Eq. (5) into Eq. (4) we can rewrite Newton’s 2\textsuperscript{nd} law as:

\[
\frac{d\mathbf{\tilde{V}}}{dt} = -2\tilde{\Omega} \times \mathbf{\tilde{V}} - \frac{1}{\rho} \nabla p + g + F_r
\]  

(6)

What do all of the terms in Eq. (6) represent? What happened to the centrifugal acceleration?

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(2 points)