Derivation of Coriolis force

To derive the zonal component, $\left(\frac{du}{dt}\right)_{co}$, consider a latitudinal or vertical displacement and utilize conservation of angular momentum.

Equate the total angular momentum at an initial distance R to a displaced distance $R + \delta R$. (if R changes u must also change).

$$\left(\Omega + \frac{u}{R}\right)R^2 = \left(\Omega + \frac{u + \delta u}{R + \delta R}\right)\left(R + \delta R\right)^2$$

- Solve for δu: Expand RHS, neglect 2nd order differentials

$$\left(\Omega + \frac{u}{R}\right)R^2 = \left(\Omega + \frac{u + \delta u}{R + \delta R}\right)\left[R^2 + 2R\delta R + \cancelto{0}{(\delta R)^2}\right]$$

$$\left(\Omega + \frac{u}{R}\right)R^2 = R^2\Omega + 2\Omega R\delta R + R^2\left(\frac{u + \delta u}{R + \delta R}\right) + 2R\delta R\left(\frac{u + \delta u}{R + \delta R}\right)$$

divide through by R^2 ....

$$\cancel{\Omega} + \frac{u}{R} = \cancel{\Omega} + \frac{2\Omega\delta R}{R} + \frac{u + \delta u}{R + \delta R} + \frac{2\delta R}{R}\left(\frac{u + \delta u}{R + \delta R}\right)$$

multiply through by $R + \delta R$ ....

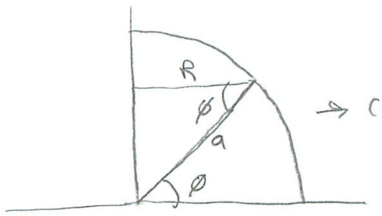
$$\cancel{u} + \frac{u\delta R}{R} = \frac{2\Omega\delta R}{R}\left(R + \delta R\right) + \cancel{u} + \delta u + \frac{2\delta R}{R}\left(u + \delta u\right)$$

$$\frac{u\delta R}{R} = 2\Omega\delta R + \cancelto{0}{\frac{2\Omega(\delta R)^2}{R}} + \delta u + \frac{2u\delta R}{R} + \cancelto{0}{\frac{2\delta R\delta u}{R}}$$

$$\frac{u\delta R}{R} = 2\Omega\,\delta R + \delta u + \frac{2u\delta R}{R}$$

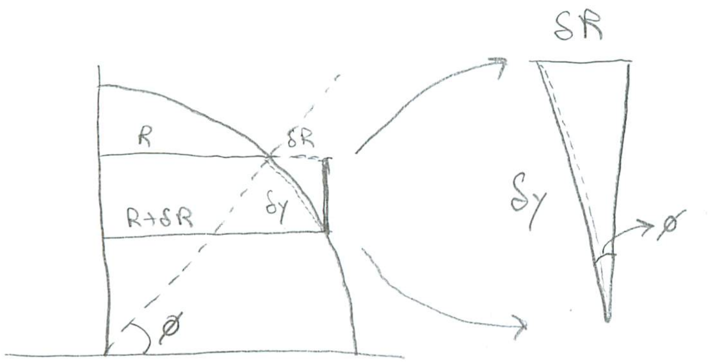
$$\delta u = -2\Omega\,\delta R - \frac{u}{R}\,\delta R$$

- note that $\boxed{R = a\cos\phi}$



$\rightarrow \cos\phi = \frac{R}{a} \rightarrow R = a\cos\phi$

- also note that $\boxed{\delta R = -\sin\phi\,\delta y}$



it can be shown that this angle $= \phi$ if we take the limit as $\delta R \rightarrow 0$

so, $\sin\phi = \frac{\delta R}{\delta y} \rightarrow \delta R = -\sin\phi\,\delta y$

substitute $R = a\cos\phi$ and $\delta R = -\sin\phi\,\delta y$ into the equation for δu, above.

$$\delta u = +2\Omega\sin\phi\,\delta y + \frac{u}{a\cos\phi}\sin\phi\,\delta y$$

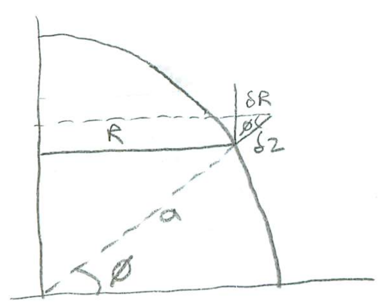
divide through by δt and take the limit as $\delta t \rightarrow 0$

$$\frac{du}{dt} = \left(2\Omega \sin\phi + \frac{u}{a}\tan\phi \right)\frac{dy}{dt}$$

substitute $\rightarrow \frac{dy}{dt} = v$

$$\boxed{\left(\frac{du}{dt}\right)_{co} = 2\Omega v \sin\phi + \frac{uv}{a}\tan\phi}$$

- For a vertical displacement $\delta R = \cos\phi\, \delta z$



$$\cos\phi = \frac{\delta R}{\delta z} \rightarrow \delta R = \cos\phi\, \delta z$$

- substitute $\delta R = \cos\phi\, \delta z$ into our previous equation for δu, divide by δt and take the limit as $\delta t \rightarrow 0$.

$$\frac{\delta u}{\delta t} = -2\Omega \cos\phi \frac{\delta z}{\delta t} - \frac{u}{a\cos\phi}\cos\phi\frac{\delta z}{\delta t}$$

using $w = \frac{dz}{dt}$ we obtain ....

$$\boxed{\left(\frac{du}{dt}\right)_{co} = -2\Omega w \cos\phi - \frac{uw}{a}}$$

* To find the meridional component of the coriolis force, $\left(\frac{dv}{dt}\right)_{co}$, consider the excess in the centrifugal force: the difference in force between an East-West moving object and an object at rest.

centrifugal force → $\frac{d\vec{v}}{dt} = \omega^2 r$

- for a moving object on Earth → $\frac{d\vec{v}}{dt} = \left(\Omega + \frac{U}{R}\right)^2 \vec{R}$
- for a still object → $\frac{d\vec{V}}{dt} = \Omega^2 \vec{R}$

subtract the two ....

$$\left(\Omega + \frac{U}{R}\right)^2 \vec{R} - \Omega^2 \vec{R}$$

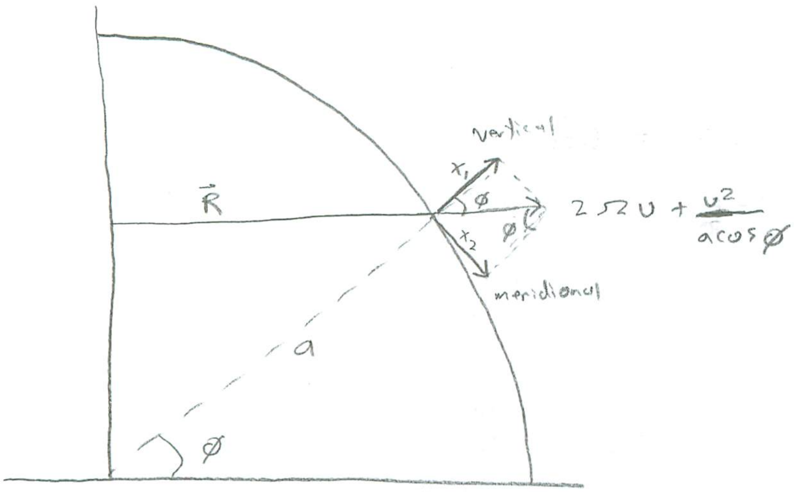
expand ....

$$\cancel{\Omega^2 \vec{R}} + \underbrace{2\Omega U \frac{\vec{R}}{R} + U^2 \frac{\vec{R}}{R^2}} - \cancel{\Omega^2 \vec{R}}$$

↳ these are "deflecting forces" and act in same direction as centrifugal force

* To find $\left(\frac{dv}{dt}\right)_{co}$ for meridional and vertical displacements, take the meridional and vertical components of $2\Omega U + \frac{U^2}{R}$.

Do them one at a time ......



the vertical component,

$$\boxed{\left(\frac{dw}{dt}\right)_{co} = 2\Omega u \cos\phi + \frac{u^2}{a}}$$

the meridional component,

$$\boxed{\left(\frac{dv}{dt}\right)_{co} = -2\Omega u \sin\phi - \frac{u^2}{a}\tan\phi}$$

For synoptic scale motions $|u| \ll \Omega R$ which means that $\frac{u^2}{a}\tan\phi$ and $\frac{uv}{a}\tan\phi$ become very small, thus we can approximate the coriolis acceleration for horizontal motion as ...

$$\left.\begin{aligned} \left(\frac{du}{dt}\right)_{co} &= 2\Omega v \sin\phi = fv \\ \left(\frac{dv}{dt}\right)_{co} &= -2\Omega u \sin\phi = -fu \end{aligned}\right\}$$

where $f = 2\Omega \sin\phi$ is the coriolis parameter.