Chapter 3: Elementary Applications of the Basic Equations

* We will actually begin to apply the equations of motion!! this is what Chapter 3 is all about.

* In chapter 2 we derived the geostrophic balance equation. There are many other approximate expressions for relationships between velocity, pressure, wind, and temperature which we will examine in Chapter 3.

* However..... These relationships are most conveniently expressed using pressure as a vertical coordinate (not height, like what we've been using). So, we first have to convert the dynamical equations to isobaric coordinates which we should get done w/ by end of week.

* 3 Equations to convert

1) horizontal momentum equation
2) continuity equation
3) thermodynamic equation

* Start w/ horizontal momentum

→ A form of the component equations we were able to arrive @ in Chapter 2 was ...

$$\left.\begin{aligned} \frac{du}{dt} - fv &= -\frac{1}{\rho}\frac{dp}{dx} \\ \frac{dv}{dt} + fu &= -\frac{1}{\rho}\frac{dp}{dy} \end{aligned}\right\}$$ how did we get this???

→ In vectorial form this is ...

$$\frac{d\vec{V}}{dt} + f\hat{k} \times \vec{V} = -\frac{1}{\rho}\nabla p$$

** note: $\vec{V} = u\hat{i} + v\hat{j}$ is the horizontal velocity vector

* To transform to isobaric coordinates ...

Recall $\frac{1}{\rho}\nabla p = \nabla_p \Phi$ where ∇_p denotes a constant pressure surface

we get ...

$$\frac{d\vec{V}}{dt} + f\vec{k} \times \vec{V} = -\nabla_p \Phi$$

* Using p as the vertical coordinate the total derivative must be expanded differently than before ...

$$\frac{d()}{dt} = \frac{\partial()}{\partial t} + \frac{dx}{dt}\frac{\partial()}{\partial x} + \frac{dy}{dt}\frac{\partial()}{\partial y} + \boxed{\frac{dp}{dt}\frac{\partial()}{\partial p}}$$

$$= \frac{\partial()}{\partial t} + u\frac{\partial()}{\partial x} + v\frac{\partial()}{\partial y} + \omega\frac{\partial()}{\partial p}$$

* Here $\omega \equiv \frac{dp}{dt}$ is referred to as the "omega vertical motion". It is the pressure change following the motion. It plays the same role in isobaric coordinates that $w = \frac{dz}{dt}$ plays in height coordinates.

* How can we express the geostrophic relationship in isobaric coordinates?

$\frac{d\vec{v}}{dt} \to 0$, take $\hat{k}\times$ of both sides of momentum equation.

* on the left side we end up with a term $\hat{k}\times(\hat{k}\times\vec{v})$

* We can use the vector identity ...

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

to show that

$$\hat{k}\times(\hat{k}\times\vec{V}) = -\vec{V}$$

thus

$$-f\vec{V}_g = -\hat{k}\times\nabla_p\Phi$$

$\boxed{f\vec{V}_g = \hat{k}\times\nabla_p\Phi}$ is the form of the geostrophic relationship in isobaric coordinates.

$\rightarrow \boxed{\vec{V}_g = \frac{1}{f}\hat{k}\times\nabla_p\Phi}$ → perpendicular to gradient in ~~pressure~~ height or parallel to ~~[illegible]~~ ~~isobars~~. iso-heights?

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* Advantages of this form over the height coordinate form ....

→ density does not appear!! thus, a given geopotential gradient @ any ~~height~~ pressure implies the same geostrophic wind, whereas a given horizontal pressure gradient implies different values depending on density.

* If f is a constant we can show that the horizontal divergence of the geostrophic wind @ constant pressure is zero.

mathematically → $\nabla_p\cdot\vec{V}_g = 0$

* To prove substitute back in geostrophic wind ---

$$\nabla_p \cdot \vec{V}_g = \nabla_p \cdot \left[\frac{1}{f}\hat{k} \times \nabla_p \Phi\right] = \frac{1}{f}\nabla_p \cdot \left[\hat{k} \times \nabla_p \Phi\right]$$

We can use another vector identify ---

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - A \cdot (\nabla \times \vec{B})$$

to expand $\nabla_p \cdot [\hat{k} \times \nabla_p \Phi]$ ---

$$\nabla_p \Phi \cdot (\cancel{\nabla_p \times \hat{k}}) - \hat{k} \cdot (\cancel{\nabla_p \times \nabla_p \Phi})$$

curl of a unit vector is zero

curl in gradient of a scalar is zero

Thus --- $\boxed{\nabla_p \cdot \vec{V}_g = 0}$

* So how do we get divergence and convergence in the real atmosphere ?? ageostrophic winds $(\vec{V} - \vec{V}_g)$

~~GO BACK to Total Derivative thing ---~~

Continuity Equation

* It is possible to transform $\frac{1}{\rho}\frac{d\rho}{dt} + \nabla\cdot\vec{V} = 0$ from height to pressure coordinates but easier to derive considering a Lagrangian control volume...

- Define controle volume as $\delta V = \delta x\,\delta y\,\delta z$

  apply the hydrostatic equation $\frac{\delta p}{\delta z} = -\rho g \rightarrow \delta z = \frac{\delta p}{-\rho g}$

- thus, $\delta V = -\delta x\,\delta y\,\frac{\delta p}{\rho g}$

- Mass is conserved following the motion and can be expressed...

$$\delta M = \rho\,\delta V$$

$$\delta M = \cancel{\rho}\left[-\delta x\,\delta y\,\frac{\delta p}{\cancel{\rho} g}\right]$$

$$\delta M = -\delta x\,\delta y\,\frac{\delta p}{g}$$

- Because mass is conserved we can write...

[per unit mass]

$$\boxed{\frac{1}{\delta M}}\frac{d}{dt}(\delta M) = \frac{\cancel{g}}{\delta x\,\delta y\,\delta p}\frac{d}{dt}\left(\frac{\delta x\,\delta y\,\delta p}{\cancel{g}}\right) = 0$$

gs cancel....

• Differentiate (use product rule)

$$\frac{1}{\delta x \delta y \delta p}\left[\delta x \delta y \frac{d\delta p}{dt} + \delta y \delta p \frac{d\delta x}{dt} + \delta x \delta p \frac{d\delta y}{dt}\right] = 0$$

• multiply through by $\frac{1}{\delta x \delta y \delta p}$ ----- Bring δ to outside...

$$\frac{1}{\delta x}\delta\left(\frac{dx}{dt}\right) + \frac{1}{\delta y}\delta\left(\frac{dy}{dt}\right) + \frac{1}{\delta p}\delta\left(\frac{dp}{dt}\right) = 0$$

• This reduces to ...

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p} = 0$$

• Take the limit as $\delta x, \delta y, \delta p \rightarrow 0$ and observe that δx and δy are evaluated at constant pressure to obtain continuity equation in isobaric system!

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$$

* Advantages ...

→ Simplicity!! no reference to time derivatives or density field.

3.1.3 Thermodynamic Energy Equation

* In chapter 2 we derived the alternative form of the thermodynamic energy equation...

$$c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = J$$

* To express in isobaric coordinates let $\frac{dp}{dt} = \omega$ and expand $\frac{dT}{dt}$ using Total derivative definition for p-coordinates. Also, substitute $\alpha = \frac{RT}{p}$ ---

$$c_p\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p}\right] - \frac{RT}{p}\omega = J$$

* divide by c_p ---

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p} - \frac{RT}{pc_p}\omega = \frac{J}{c_p}$$

factor out $-\omega$

and define $S_p \equiv \frac{RT}{c_p p} - \frac{\partial T}{\partial p}$

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - S_p\omega = \frac{J}{c_p}$$

S_p is the stability parameter for the isobaric system $\rightarrow$ we can show this using the equation of state and poisson's equation.

* Poisson's equation ....

$$\theta = T\left(\frac{p_s}{p}\right)^{R/c_p}$$

* take the natural log and differentiate with respect to pressure...

$$\ln\theta = \ln T + \frac{R}{c_p}\cancel{\ln p_s} - \frac{R}{c_p}\ln p$$

$$\underset{0}{}$$

$$\frac{1}{\theta}\frac{\partial\theta}{\partial p} = \frac{1}{T}\frac{\partial T}{\partial p} - \frac{R}{c_p p} \qquad \text{multiply by } T\ ---$$

$$\frac{T}{\theta}\frac{\partial\theta}{\partial p} = \frac{\partial T}{\partial p} - \frac{RT}{c_p p} \qquad \text{multiply by } -1$$

$$-\frac{T}{\theta}\frac{\partial\theta}{\partial p} = \frac{RT}{c_p p} - \frac{\partial T}{\partial p} \longrightarrow \text{Recall, } S_p \equiv \frac{RT}{c_p p} - \frac{\partial T}{\partial p}$$

thus ...

$$S_p = -\frac{T}{\theta}\frac{\partial\theta}{\partial p}$$

* We can express this in terms of the dry adiabatic and environmental lapse rates as we did before.

use chain rule ----

$$-\frac{T}{\theta}\frac{\partial\theta}{\partial p} = -\frac{T}{\theta}\frac{\partial\theta}{\partial z}\frac{\partial z}{\partial p}$$

from chapter 2 $\rightarrow \frac{T}{\theta}\frac{\partial\theta}{\partial z} = \Gamma_d - \Gamma$

hydrostatic equation $\longrightarrow \frac{\partial z}{\partial p} = \frac{1}{-\rho g}$

$$-\frac{T}{\theta}\frac{\partial\theta}{\partial p} = (\Gamma_d - \Gamma)/\rho g$$

* As long as the environmental lapse rate is less than the dry adiabatic lapse rate

S_p is positive → stable

* Because ρ decreases exponentially w/ height S_p increases rapidly. This strong height dependence is a minor disadvantage of isobaric coordinates.