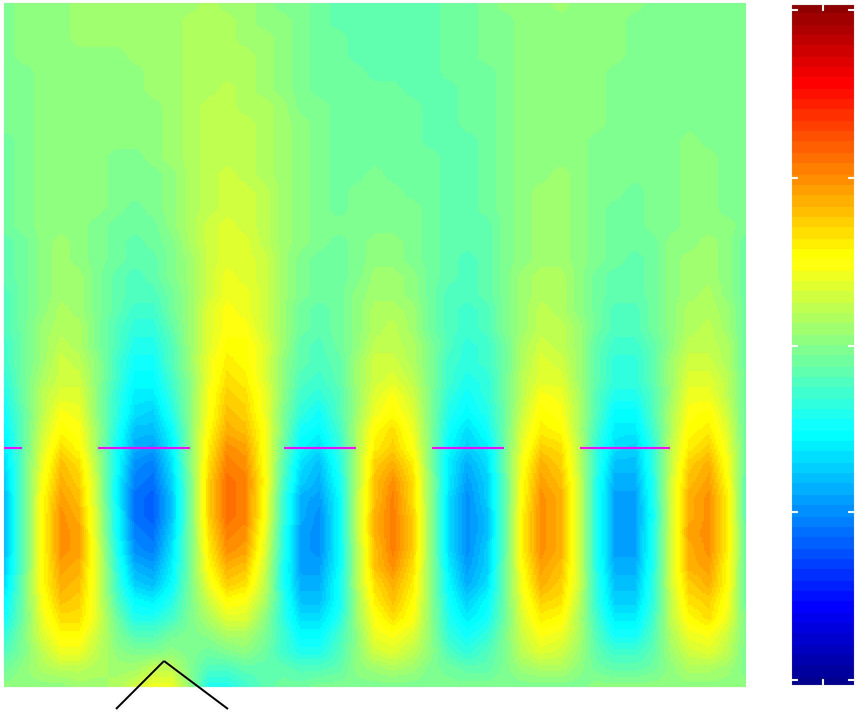
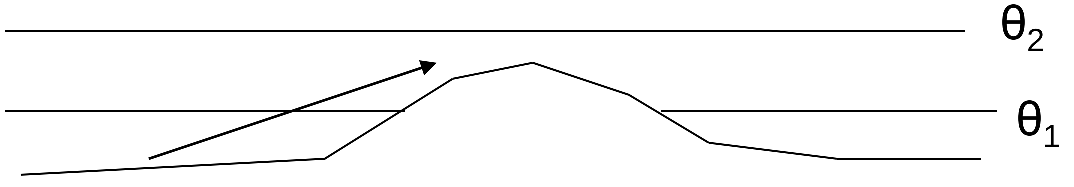
Mountain wave dynamics



- Consider stratified flow past an isolated ridge



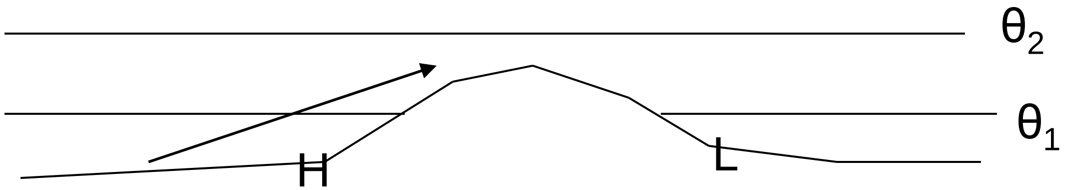
•From its origin to the ridge top, the parcel's buoyancy is negative, so that its motion is inhibited. But after it crosses the ridge top, the negative buoyancy acts to accelerate the parcel as it travels downhill.

•So we can expect:

-- Low wind speeds on the windward side of the ridge

-- Higher wind speeds on the lee side

- This results in…



•Accumulation of the colder (Denser) fluid on the windward side → hydrostatic pressure increase

•Divergence → hydrostatic pressure decrease on the lee side

- This pressure pattern implies two consequences:

-- there is a “wave drag” exerted by the atmospheric flow on the earth, or equivalently the mountain exerts a drag on the atmosphere (in the opposite direction)

-- if the flow is in steady-state, then the wave drag must be exactly balanced by a downward flux of momentum

-- this downward momentum flux must be constant with height; otherwise we would get an acceleration (deceleration) of the flow where we had a vertical flux convergence (divergence)

- We can write the momentum flux at a point as $\rho u'w'$
- To get the total momentum flux exerted on the mountain we have to integrate from one side of the mountain to the other

$$M=\int_{-\infty}^{\infty} \rho u'w' \, dx$$

- This is balanced by the pressure force exerted on the mountain. We can get a solution for the pressure force (wave drag) if we make some assumptions about the shape of the ridge.

Isolated (non-periodic) terrain

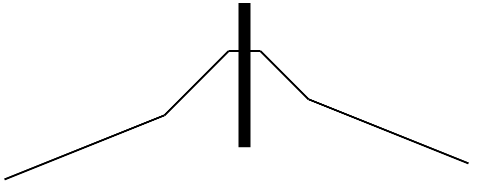
- For periodic terrain, it is often assumed that the terrain is sinusoidal. For **isolated** terrain, the ridge is often assumed to have a bell-shaped profile. It is convenient to specify this profile as a function called the Witch of Agnesi:

$$h(x) = h_0[a^2/(x^2+a^2)]$$

Where h_0 = ridgetop height

x = distance from ridge centerline

a = characteristic length scale



- The length “a” is the half-height width (or sometimes just called the “half-width”) of the terrain; i.e., $h=(1/2)\,h_0$ at a distance $x=+/-$ a from the ridge centerline.
- The function is also useful because it has a simple Fourier transform:

$$h(k) = 1/\pi \int_{-\infty}^{\infty} h(x)\, e^{-ikx}\, dx = h_0 a e^{-ka}$$

Which implies that the dominant horizontal wavelength is given by $k = 1/a$ or

$L_x = 2\pi a$

- For the Witch of Agnessi the wave drag can be solved as:

$D = \pi/4 \, \rho_0 \, N \, U \, h_0^2$

We see…

D is proportional to density; a denser fluid exerts more drag than a lighter one (water or molasses compaired to air)

D is proportional to the stratification: as the flow tends to neutral stratification, N $\rightarrow$ 0 and there is no wave drag. This is because there are no buoyancy perturbations.

D is proportional to the wind speed

D is proportional to the square of the ridge height.

Mountain waves

- There is a classical linear solution for mountain waves, which is worth exploring in some depth because it tells us a lot about how mountain waves act.
- We begin with a linearized set of equations for an atmosphere with a horizontally-uniform basic state U, θ_0, which may vary with height. We further assume that the flow is uniform in the transverse (y) direction, and is in steady state ($\partial/\partial t = 0$).

- U-momentum:

  $U\partial u'/\partial x + w\partial U/\partial z + \partial p'/\partial x = 0$

- W-momentum:

  $U\partial w/\partial x + \partial p'/\partial z = b$, where $b=g\theta'/\theta_0$ is the buoyant acceleration.

- Thermodynamic equation:

  $U\partial b/\partial x + wN^2 = 0$

- The thermo eqn looks odd. We can recognize $U\partial b/\partial x$ as having the usual form for the (horizontal) advection. But what about wN^2?
- General rule: if a term looks odd, expand it out and try to find things you recognize. So recall from basic dynamics that $N=(g/\theta\,\partial\theta/\partial z)^{1/2}$ is the Brunt-Vaisala freq. Then,

$$wN^2 = w\,(g/\theta)\partial\theta/\partial z = g/\theta(w\partial\theta/\partial z)$$

We can now recognize $(w\partial\theta/\partial z)$ as the vertical advection of potential temperature.

In essence, the thermo eqn is just an advection eqn for the potential temperature

$$U\partial\theta'/\partial x + w\partial\theta_0/\partial z = 0$$

Multiply by g/θ_0 to recast things in terms of buoyancy:

$$Ug/\theta_0\partial\theta'/\partial x + wg/\theta_0\partial\theta_0/\partial z = 0$$

$\partial b/\partial x$ $\qquad\qquad$ N^2

Continuity equation:

$$\partial u'/\partial x + \partial w/\partial z = 0$$

Question: Is this system of equations hydrostatic? (where does the eq. for hydrostatic balance come from?)

Derivation of mountain wave equations

- Start with 4 equations…

$$U\partial u'/\partial x + w\partial U/\partial z + \partial p'/\partial x = 0$$

$$U\partial w/\partial x + \partial p'/\partial z = b$$

$$U\partial b/\partial x + wN^2 = 0$$

$$\partial u'/\partial x + \partial w/\partial z = 0$$

Take $\partial/\partial z$ (1) and $\partial/\partial x$ (2):

$\partial U/\partial z \partial u'/\partial x + U\partial^2 u'/\partial x\partial z + \partial w/\partial z \partial U/\partial z + w\partial^2 U/\partial z^2 + \partial^2 p'/\partial x\partial z = 0$ (from 1 → 5)

$U\partial^2 w/\partial x^2 + \partial^2 p'/\partial x\partial z = \partial b/\partial x$ (2 → 6)

Divide (3) by U:

$\partial b/\partial x = -wN^2/U$ (7) and substitute into (6):

$U\partial^2 w/\partial x^2 + \partial^2 p'/\partial x\partial z + wN^2/U = 0$ (8)

Solve (8) for $\partial^2 p'/\partial x\partial z = - U\partial^2 w/\partial x^2 - wN^2/U$ (9)

And substitute into 5:

$\cancel{\partial U/\partial z \partial u'/\partial x} + U\partial^2 u'/\partial x\partial z + \cancel{\partial w/\partial z \partial U/\partial z} + w\partial^2 U/\partial z^2 - U\partial^2 w/\partial x^2 - wN^2/U = 0 \quad (10)$

Since $\partial u'/\partial x = -\partial w/\partial z$ by our continuity eqn (4), the 1st and 3rd terms cancel out.

$U\partial^2 u'/\partial x\partial z = U\partial/\partial z(\partial u'/\partial x) = -U\partial^2 w/\partial z^2$

So we can rewrite (10) as:

$-U\partial^2 w/\partial z^2 + w\partial^2 U/\partial z^2 - U\partial^2 w/\partial x^2 - wN^2/U = 0$

Multiply by -1/U to get

$\partial^2 w/\partial z^2 + \partial^2 w/\partial x^2 + (N^2/U^2 – 1/U\ \partial^2 U/\partial z^2)w = 0$

Consider the units of term in parenthesis. They should be m^{-2} since N^2 is in s^{-2} and U^2 is in m^2s^{-2}.

We can then define $(N^2/U^2 – 1/U\ \partial^2 U/\partial z^2) = \ell^2$

Where ℓ=vertical wavenumber, and ℓ^2 is the Scorer parameter.

- Thus, we have a second order diffeq → wave-type solutions are of form:
- $w(x,z) = w_1(z)\cos(kx) + w_2(z)\sin(kx)$

where k is the horiz. Wavenumber. Our lower boundary condition implies k is the wavenumber of the underlying terrain → that is the shape of periodic ridges and valleys.

- Look at our equation. We have $\partial^2 w/\partial z^2$ and $\partial^2 w/\partial x^2$. We can take $\partial^2/\partial x^2$ of this solution (a trick for 2nd derivative sin or cosine):

$$\partial^2 w/\partial x^2 = \hat{w}_1(z)k^2\cos(kx) - k^2\hat{w}_1(z)\sin(kx)$$

$$= -k^2[\hat{w}_1(z)\cos(kx) + \hat{w}_1(z)\sin(kx)]$$

$$= -k^2\, w(x,z)$$

And plug back into the 2nd order diffeq to get:

$\partial^2 w/\partial z^2 - k^2 w + \ell^2 w = 0$

$\partial^2 w/\partial z^2 + (\ell^2 - k2)w = 0$

Divide the whole eqn by [cos(kx) or sin(kx)] to get:

$\partial^2 \widehat{w}_i/\partial z^2 + (\ell^2 - k^2)\hat{w}_i = 0$ where i=1,2

→K is fixed by the terrain

→If we assume N, U constant, then ℓ = N/U = constant

→Define $m^2 = \ell^2 - k^2$; $\mu^2 = -m^2$

- Then the solution of the vertical part depends on the relation between k^2 and ℓ^2:

(1) For $k > \ell$, then $(\ell^2 - k^2)^{1/2}$ is complex so

$\widehat{w}_i(z) = Ae^{\mu z} + Be^{-\mu z}$

(2) For $k < \ell$, then $(\ell^2 - k^2)^{1/2}$ is real and

$\hat{w}_i(z) = A\cos(mz) + B\sin(mz)$

This says a lot about the vertical structure of the mountain wave:

- For $k>\ell$ we see that the amplitude w decreases exponentially with height (evanescent). The other solution $e^{\mu z}$ is discarded as nonphysical (increases without bound)
- For $k<\ell$ the phase oscillates with height, i.e. at some height above the terrain, the phase reverses and then the original phase is recovered as we go higher. The amplitude is constant.
- Consider the physical meaning of $k<\ell$….(broad waves). Also note that $U/Na \ll 1$with these broad waves yields hydrostatic waves, $U/Na \approx 1$ yields nonhydrostatic waves.

- Notice that m gives the vertical wavelength. For long horizontal waves, we have $k \ll \ell$ so that $m \approx \ell$. Then,

$$m \approx \ell = (N^2/U^2 - 1/U\ \partial^2 u/\partial z^2)^{1/2}$$

If we assume the background flow is constant with height, then $m \approx N/U$

so that the vertical wavelength is

$$L_z = 2\pi/m \approx 2\pi U/N$$

Typical values: assume U=10 m/s, $\partial\theta/\partial z \sim 3.3\times10^{-3}$ Km^{-1}, then $N=(g/\theta\,\partial\theta/\partial z)^{1/2} \approx 0.01\ s^{-1}$, so

$$L_z \approx 2\pi(10 m/s)/10^{-2}\ s^{-1} \approx 6000\ m.$$

- Notice in general, higher windspeeds → greater vertical wavelength

greater stability → smaller vertical wavelength.

Recall that we derived a linear solution. One condition for linearity is that the amplitude of the vertical displacement (i.e. height of the mountain) must be much less than the vertical wavelength:

$h \ll 2\pi U/N$ or $Nh/U \ll 2\pi$, which is often simplified to $NH/U \ll 1$

Hydrostatic Mountain waves

- $K^2 << \ell^2$ where k is the horizontal wavenumber and l is the vertical wavenumber

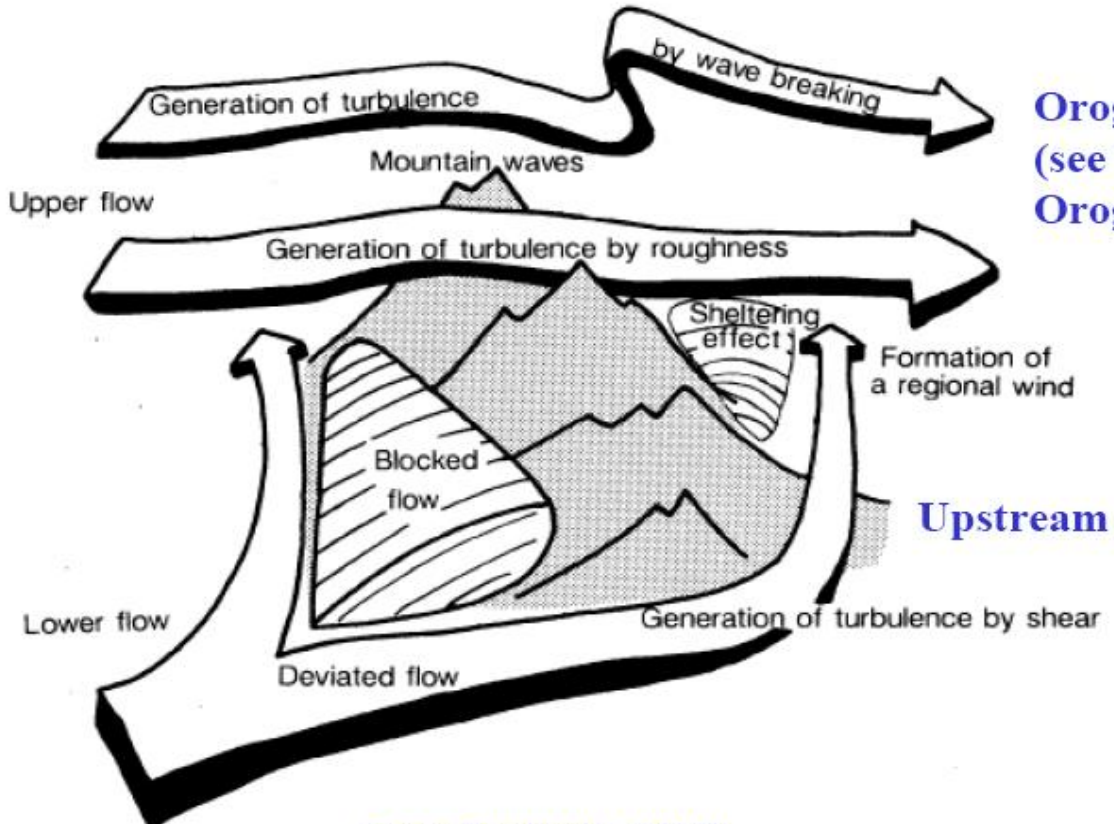
$(2\pi/L_x)^2 << N^2/U^2$

Define a characteristic width $a = L_x/2\pi$ (witch of Agnessi), then

$1/a^2 << N^2/U^2$ or $U/Na << 1$.

In terms of scaling, this means that the time scale for flow to cross the mountain, or the lagrangian time scale $\tau_L = a/U$ is long compared to the period of a buoyant oscillation, $\tau_B = 1/N$

Mountain waves = gravity waves = buoyancy waves



Orographic turbulence (see Turbulent Orographic Form Drag)

Upstream / low-level blocking

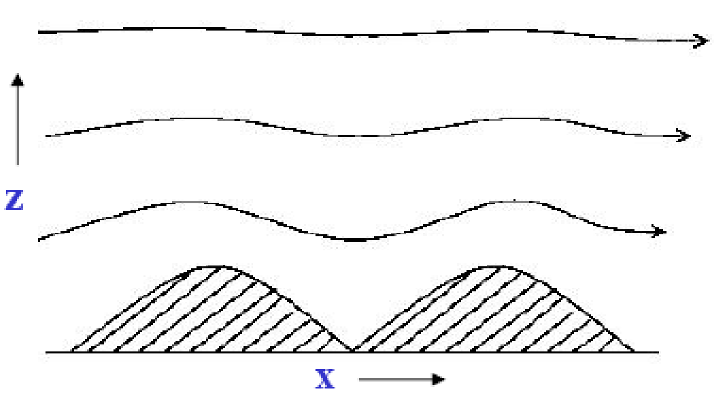
From Bougeault (1990)

Evanescent solution (i.e. fading away)

$$w = \mathrm{Re}\left\{e^{ikx}\left(w_1 e^{-|\mu|z} + \cancelto{0}{w_2 e^{|\mu|z}}\right)\right\}$$
$$m = \sqrt{l^2 - k^2} = i\mu$$

- waves decay exponentially with height
- flow stays in phase with mountain
- no momentum transport

$$w = Ae^{-|\mu|z}\sin kx$$



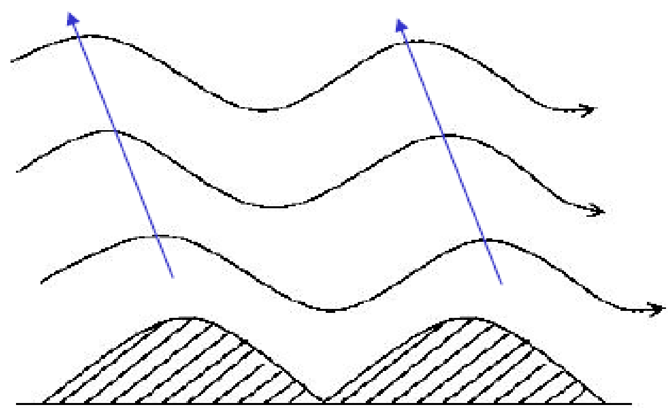
K > ℓ

Wave solution

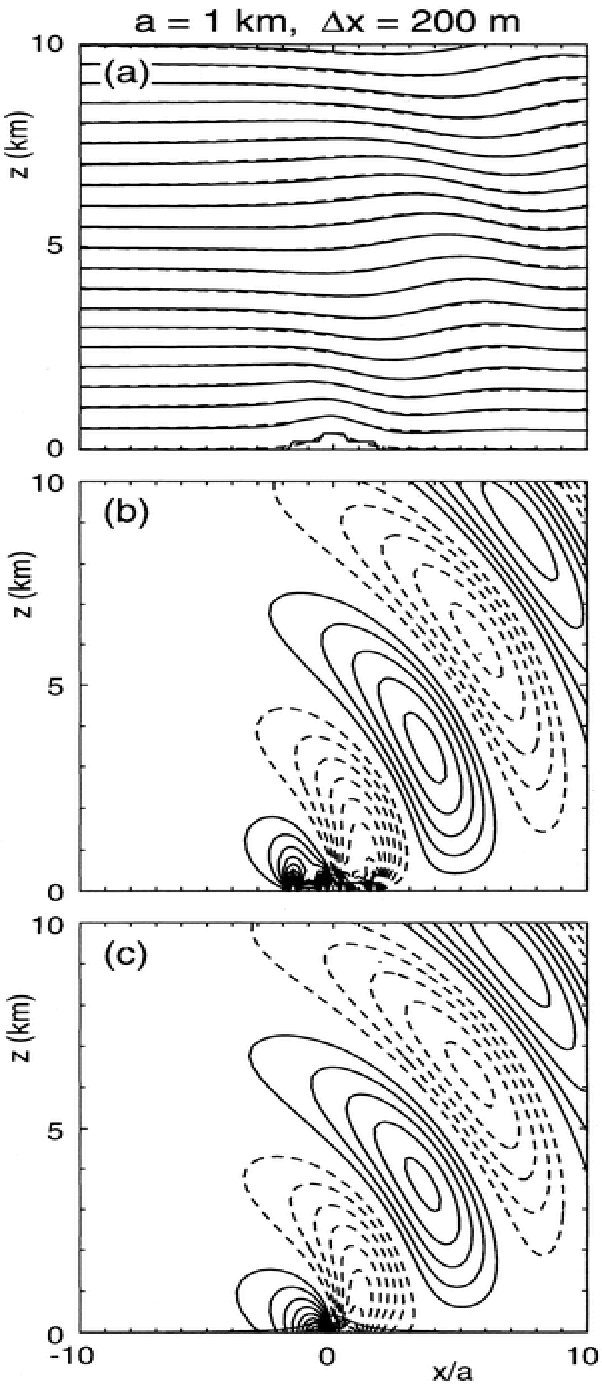
$$w = \mathrm{Re}\left\{w_1 e^{i(kx+mz)} + \cancelto{0}{w_2 e^{i(kx-mz)}}\right\}$$
$$m = \sqrt{l^2 - k^2} > 0$$

- energy/momentum transported upwards
- waves propagate without loss of amplitude
- phase lines tilt upstream as z increases

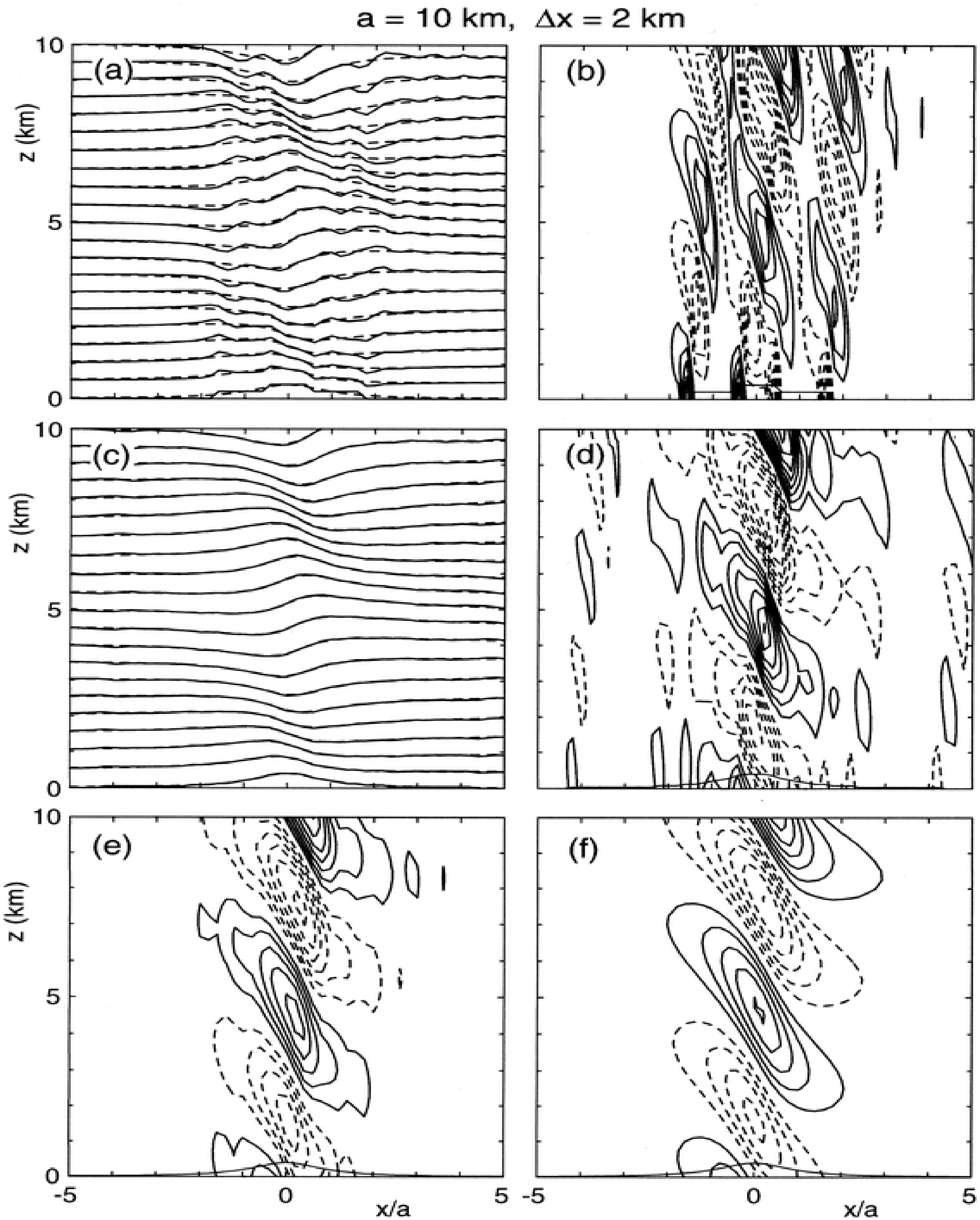
$$w = A\sin(kx + mz)$$



K < ℓ

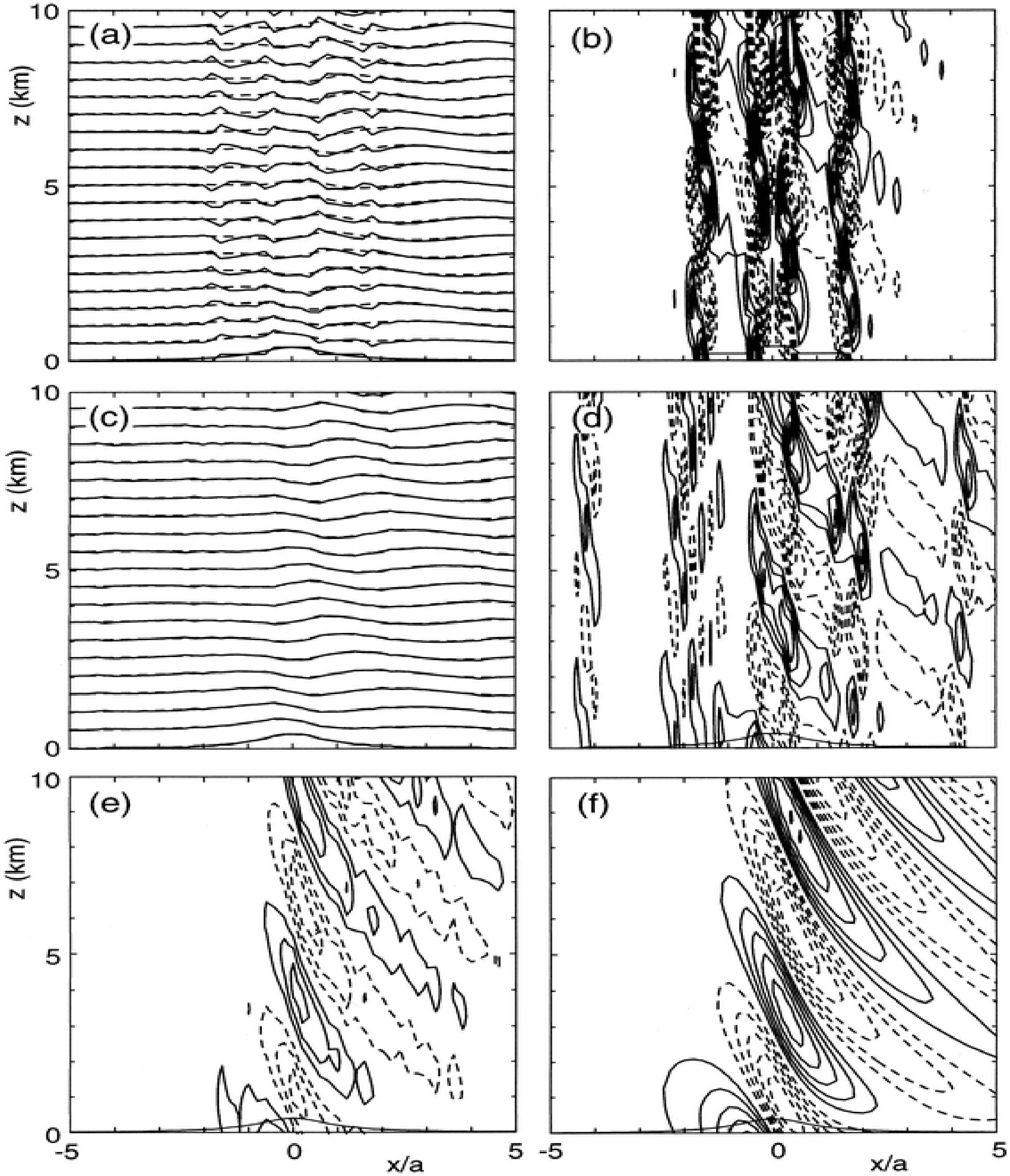


a = 1km



a=10km

a =100 km, Δx = 20 km



Interpretation of figures

- These figures are taken from a test of a model that shows how flow varies over mountains of differing widths (Gallus and Rancic 1996).
- Note that for smallest width, waves damp out with height, but for bigger widths, they do not.
- For widths around 1km, we get nonhydrostatic mountain waves that seem to propagate downstream and upward (THIS IS A GOOD TEST OF A NONHYDROSTATIC MODEL – IT SHOULD BE ABLE TO SHOW THIS)
- As mountain gets broader, hydrostatic waves are present and seem stacked above mountain
- When mountain gets very broad (100 km), it takes enough time for flow to cross the mountain that the Coriolis force acts to produce these new types of behavior