407 Closure

Averaging the governing equations: the closure problem.

When we solve the eqs. in a numerical model we are really solving for averages. There has been a great deal of discussion on the philosophy and consequences of this averaging. Briefly, there are three main types of averages:

- spatial average: typically interpreted as the average over a grid volume.
- temporal average: average over some period of time; e.g. a day or a time step.
- ensemble average: average over numerous realizations of some broadly similar circumstance.

In mesoscale models the grid-point values are generally interpreted as averages over the grid volume and over the time step: i.e., for some dependent variable α, we have

$$\bar{\alpha}(x,y,z,t) = \frac{1}{\Delta x\, \Delta y\, \Delta z\, \Delta t} \int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} \int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} \int_{z-\frac{\Delta z}{2}}^{z+\frac{\Delta z}{2}} \alpha(x,y,z,t)\, dz\, dy\, dx\, dt$$

We now need to define (or assume) some characteristics of the averaging procedure:

- Any variable α at any point in space-time can be expressed as a mean and a perturbation: $\alpha = \bar{\alpha} + \alpha'$
- The average of the deviations is zero:

  $$\overline{\alpha'} \equiv 0$$

  (This is the Reynolds assumption.)
- The statistics are stationary over the averaging interval; i.e. the mean is constant. Then it follows that

  $$\overline{\overline{\alpha}} = \overline{\alpha}$$

- From the preceding we see that:

  since $\overline{\alpha'} = 0$, $\overline{c\alpha'} = 0$ for some constant c

  thus $\overline{\alpha'\bar{\alpha}} = 0$. (since $\bar{\alpha}$ is a constant)

  Further,

  $$\overline{\frac{\partial \alpha}{\partial x_i}} = \frac{\partial}{\partial x_i}\overline{\alpha}$$

- for two variables α and β,

  $$\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}$$

- but notice that in general,

  $$\overline{\alpha\beta} = \overline{(\bar{\alpha} + \alpha')(\bar{\beta} + \beta')} \neq \bar{\alpha}\bar{\beta} \text{ ; that is,}$$

  $$\begin{aligned} \overline{\alpha\beta} &= \overline{\bar{\alpha}\bar{\beta}} + \overline{\bar{\alpha}\beta'} + \overline{\alpha'\bar{\beta}} + \overline{\alpha'\beta'} \\ &= \bar{\alpha}\bar{\beta} + \overline{\alpha'\beta'} \end{aligned}$$

Now let us look at the consequences of the averaging. Consider the thermodynamic eq:

$$\frac{\partial \theta}{\partial t} = -u\frac{\partial \theta}{\partial x} - v\frac{\partial \theta}{\partial y} - w\frac{\partial \theta}{\partial z} + \dot{Q}$$

where Q is the diabatic heating rate.

We decompose θ into a mean and a perturbation:

$$\theta = \bar{\theta} + \theta'$$

and likewise for the other variables,

$$u_i = \bar{u}_i + u_i' \;; \quad Q = \bar{Q} + Q'$$

Then the thermo eq. looks like:

$$\frac{\partial}{\partial t}(\bar{\theta}+\theta') = -(\bar{u}+u')\frac{\partial}{\partial x}(\bar{\theta}+\theta') - (\bar{v}+v')\frac{\partial}{\partial y}(\bar{\theta}+\theta') - (\bar{w}+w')\frac{\partial}{\partial z}(\bar{\theta}+\theta') + (\bar{Q}+Q')$$

<u>Average</u> the equation:

$$\frac{\partial}{\partial t}(\bar{\theta}) + \frac{\partial}{\partial t}\cancelto{0}{(\overline{\theta'})} = -\left(\bar{u}\frac{\partial\bar{\theta}}{\partial x} + \cancelto{0}{\overline{u'\frac{\partial\bar{\theta}}{\partial x}}} + \cancelto{0}{\overline{\bar{u}\frac{\partial\theta'}{\partial x}}} + \overline{u'\frac{\partial\theta'}{\partial x}}\right)$$
$$-\left(\bar{v}\frac{\partial\bar{\theta}}{\partial y} + \cancel{\overline{v'\frac{\partial\bar{\theta}}{\partial y}}} + \cancel{\overline{\bar{v}\frac{\partial\theta'}{\partial y}}} + \overline{v'\frac{\partial\theta'}{\partial y}}\right)$$
$$-\left(\bar{w}\frac{\partial\bar{\theta}}{\partial z} + \cancel{\overline{w'\frac{\partial\bar{\theta}}{\partial z}}} + \cancel{\overline{\bar{w}\frac{\partial\theta'}{\partial z}}} + \overline{w'\frac{\partial\theta'}{\partial y}}\right)$$
$$+\ \bar{Q} + \cancelto{0}{\overline{Q'}}$$

Result is:

$$\frac{\partial \bar{\theta}}{\partial t} = -\bar{u}\frac{\partial \bar{\theta}}{\partial x} - \bar{v}\frac{\partial \bar{\theta}}{\partial y} - \bar{w}\frac{\partial \bar{\theta}}{\partial z}$$
$$- \overline{u'\frac{\partial \theta'}{\partial x} - v'\frac{\partial \theta'}{\partial y} - w'\frac{\partial \theta'}{\partial z}}$$
$$+ \bar{Q}$$

First three terms are advection of the mean potential temperature by the mean wind.

Second three terms are the <u>net</u> (mean) contribution from advection of potential temperature <u>fluctuations</u> by the <u>fluctuating part</u> of the wind. These fluctuations are often referred to as "turbulence" but this is not strictly correct: they include <u>all</u> fluctuations on scales smaller than the grid volume. We will look at these in detail momentarily.

The final term is the grid-average diabatic heating. We assume here that the diabatic heating is uncorrelated with the potential temperature; i.e., $\overline{Q'\theta'} = 0$. Therefore the positive and negative deviations of Q simply cancel one another.

Now look at the subgrid transport terms: $\overline{u' \frac{\partial u'}{\partial x}}$

By the product rule of differentiation we have:

$$\overline{u' \frac{\partial \theta'}{\partial x}} = \frac{\partial}{\partial x} (\overline{u'\theta'}) - \overline{\theta' \frac{\partial u'}{\partial x}}$$

(just a rearrangement of $\frac{\partial}{\partial x} (\overline{u'\theta'}) = \overline{u' \frac{\partial \theta'}{\partial x}} + \overline{\theta' \frac{\partial u'}{\partial x}}$).

In a similar manner,

$$\overline{v' \frac{\partial \theta'}{\partial y}} = \frac{\partial}{\partial y} (\overline{v'\theta'}) - \overline{\theta' \frac{\partial v'}{\partial y}}$$

and

$$\overline{w' \frac{\partial \theta'}{\partial z}} = \frac{\partial}{\partial z} (\overline{w'\theta'}) - \overline{\theta' \frac{\partial w'}{\partial z}}$$

Then our eq. becomes:

$$\begin{aligned} \frac{\partial \bar{\theta}}{\partial t} = & - \bar{u} \frac{\partial \bar{\theta}}{\partial x} - \bar{v} \frac{\partial \bar{\theta}}{\partial y} - \bar{w} \frac{\partial \bar{\theta}}{\partial z} \\ & - \frac{\partial}{\partial x} (\overline{u'\theta'}) - \frac{\partial}{\partial y} (\overline{v'\theta'}) - \frac{\partial}{\partial z} (\overline{w'\theta'}) \\ & - \overline{\theta' \frac{\partial u'}{\partial x}} - \overline{\theta' \frac{\partial v'}{\partial y}} - \overline{\theta' \frac{\partial w'}{\partial z}} \\ & + \bar{Q} \end{aligned}$$

The second three terms are now recognizable as "turbulence" flux divergences (or more correctly, <u>subgrid</u> flux divergences).

What about the <u>third</u> set of terms? This can be rewritten as:

$$-\overline{\left(\theta'\frac{\partial u'}{\partial x} + \theta'\frac{\partial v'}{\partial y} + \theta'\frac{\partial w'}{\partial z}\right)}$$

$$= -\overline{\theta'\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)}$$

If we assume the fluctuations obey incompressible continuity, then at every point we have

$$\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) = 0$$

so our average becomes

$$= \overline{-\theta'(0)} = 0.$$

Then our governing eqs. become:

$$\frac{\partial \bar{u}}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} + f\bar{v} - \bar{u}\frac{\partial \bar{u}}{\partial x} - \bar{v}\frac{\partial \bar{u}}{\partial y} - \bar{w}\frac{\partial \bar{u}}{\partial z}$$
$$- \frac{\partial}{\partial x}(\overline{u'u'}) - \frac{\partial}{\partial y}(\overline{u'v'}) - \frac{\partial}{\partial z}(\overline{u'w'})$$

$$\frac{\partial \bar{v}}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial y} - f\bar{u} - \bar{u}\frac{\partial \bar{v}}{\partial x} - \bar{v}\frac{\partial \bar{v}}{\partial y} - \bar{w}\frac{\partial \bar{v}}{\partial z}$$
$$- \frac{\partial}{\partial x}(\overline{u'v'}) - \frac{\partial}{\partial y}(\overline{v'v'}) - \frac{\partial}{\partial z}(\overline{v'w'})$$

$$\frac{\partial \bar{\theta}}{\partial t} = -\bar{u}\frac{\partial \bar{\theta}}{\partial x} - \bar{v}\frac{\partial \bar{\theta}}{\partial y} - \bar{w}\frac{\partial \bar{\theta}}{\partial z}$$
$$- \frac{\partial}{\partial x}(\overline{\theta'u'}) - \frac{\partial}{\partial y}(\overline{\theta'v'}) - \frac{\partial}{\partial z}(\overline{\theta'w'})$$
$$+ \bar{Q}$$

$$\frac{\partial \bar{w}}{\partial z} = -\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right)$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0$$

$$\bar{\theta} = \bar{T}\left(\frac{1000}{\bar{p}}\right)^{R_d/c_p}$$

$$\bar{p} = \bar{\rho} R_d \bar{T}$$

7 eqs.

7 first-moment variables $\bar{u}, \bar{v}, \bar{w}, \bar{\theta}, \bar{p}, \bar{\rho}, \bar{T}$

<u>but</u> we have also picked up <u>8</u> second-moment unknowns from the averaging procedure!

We might think we could make predictive eqs. for the second moments, and in fact we can. But the averaging procedure will give us additional terms involving the third moments; e.g., the predictive equation for $\overline{u'\theta'}$ works out to:

$$\frac{\partial}{\partial t}(\overline{u'\theta'}) = -\bar{u}\frac{\partial}{\partial x}(\overline{u'\theta'}) - \bar{v}\frac{\partial}{\partial y}(\overline{u'\theta'}) - \bar{w}\frac{\partial}{\partial z}(\overline{u'\theta'})$$

$$\boxed{-\frac{\partial}{\partial x}(\overline{u'u'\theta'}) - \frac{\partial}{\partial y}(\overline{v'u'\theta'}) - \frac{\partial}{\partial z}(\overline{w'u'\theta'})}$$

$$+ \text{(other terms)}$$

Clearly we can't close the system by adding eqs. for the higher-order terms, since in an eq. for the n^{th}-order moment we always add terms in the $(n+1)^{th}$-order moment.

To obtain a closed system (i.e., same number of eqs. and unknowns) we have to do something about these higher-order terms: in general, for a given term we can either

- ignore it; or
- represent the higher-order variables as functions of the lower-order variables.

First-order closure

The "order" of closure refers to the highest-order moments that appear in the governing eqs.

- Zero-order closure does not have any predictive equations.
- First-order closure has equations for the first-order moments ($\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{\theta}$, etc.) and represents the higher-order moments as a function of the first-order moments (or ignores them).
- Second-order closure includes equations for the second-order moments, e.g., $\frac{\partial}{\partial t}(\overline{w'\theta'})$ etc. The third-order moments are then represented as functions of the first and second-order moments.
- and so on for 3^{rd}, 4^{th} ... order

The highest order of closure used in a mesoscale model of which I am aware is the third-order closure sea breeze model used by Briere (1987, J. Atmos. Sci.).

By far the most common approach to first-order closure involves <u>gradient transfer theory</u>. This approach represents fluxes as proportional to gradients of the mean variables:

$$\overline{u_i' \alpha'} = -K_i \frac{\partial \overline{\alpha}}{\partial x_i}$$

That is, the net (mean) subgrid flux in a given direction is proportional to the gradient in that direction. Notice:

- the transport is "<u>down the gradient</u>", i.e., from high values of α to low values of α
- the transport depends on the <u>local</u> gradient $\partial\alpha/\partial x_i$
- the K_i's may be <u>anisotropic</u>; i.e., we could have different K's for horizontal and vertical transport.

The closure problem then reduces to specification of the K's. Common methods include

- <u>profile methods:</u> specify some particular function that gives a realistic distribution of K with height. Several such profiles have been formulated. The general characteristic is that K is small near the ground, large in the lower half of the PBL, then small again near the PBL top.

- <u>local stability / mixing length:</u> the K's are specified by some functional form so that they are small where the atmosphere is stable and increase as the atmosphere becomes less stable (or unstable). Typically $K = f(Ri)$. Also specify a <u>length scale</u> or "mixing length". Typically a "law of the wall" type approach is used so that $l \to 0$ at the ground:
$$l = kz$$
where k (little k) is von Karman's constant ($k \approx 0.40$). An upper bound l_0 is placed on k so that $l \le l_0 \approx 50 - 100$ m or so (either as a max or transition; $\frac{1}{l} = \frac{1}{kz} + \frac{1}{l_0}$).

A widely-recognized deficiency of the gradient transfer approach is that sometimes the fluxes do not depend on the local gradient. In particular, convective eddies in the daytime mixed layer transport heat across the entire PBL depth (from the ground to the top of the PBL). This transport is related more to the bulk PBL properties than to the local gradients.

Several methods have been proposed to include this nonlocal transport. One technique that is coming into fairly wide use is a simple modification of gradient transfer:

$$\overline{w'\theta'} = -K \left(\frac{\partial \bar{\theta}}{\partial z} - \gamma \right)$$

Here we can interpret $\overline{w'\theta'}$ as the sum of the local contribution $-K \frac{\partial \bar{\theta}}{\partial z}$ and the nonlocal contribution $-K\gamma$. Typically $\gamma \approx 0.5 - 1$ K km^{-1}. Recent formulations express γ essentially as a function of the surface heat flux.