Mteor 407 – Numerical methods

A fairly simple example of a numerical stability analysis can be done for the coriolis terms.
We can first look at the case of pure forward differencing:

$$\frac{\partial u}{\partial t} = fv \quad (+\text{ other terms})$$
$$\frac{\partial v}{\partial t} = -fu \quad (+\text{ other terms})$$

$$u(t+\Delta t) = u(t) + \Delta t\, f\, v(t)$$
$$v(t+\Delta t) = v(t) - \Delta t\, f\, u(t)$$

Here the velocity components for the future time (timestep number $n+1$) is simply the present value updated by the coriolis force acting on the present-value velocity.

Now we look at one fourier component of the u and v fields. We can express u as

$$u = \hat{u}\, e^{i\omega t}$$
$$v = \hat{v}\, e^{i\omega t}$$

Any time t in the future can be expressed as a certain number of time steps from the present. Then at the n^{th} time step, we have

$$u(t) = \hat{u}\, e^{i\omega(n\Delta t)}$$
$$v(t) = \hat{v}\, e^{i\omega(n\Delta t)}$$

Likewise

$$u(t+\Delta t) = \hat{u}\, e^{i\omega(n+1)\Delta t}$$
$$v(t+\Delta t) \quad \hat{v}\, e^{i\omega(n+1)\Delta t}$$

Then we can rewrite our finite-difference eqs. as:

$$\hat{u}\, e^{i\omega(n+1)\Delta t} = \hat{u}\, e^{i\omega n\Delta t} + \Delta t\, f\, \hat{v}\, e^{i\omega n \Delta t}$$
$$\hat{v}\, e^{i\omega(n+1)\Delta t} = \hat{v}\, e^{i\omega n\Delta t} - \Delta t\, f\, \hat{u}\, e^{i\omega n \Delta t}$$

We can divide both eqs by $e^{i\omega n\Delta t}$ to get:

$$\hat{u}\, e^{i\omega \Delta t} = \hat{u} + \Delta t\, f\, \hat{v}$$
$$\hat{v}\, e^{i\omega \Delta t} = \hat{v} - \Delta t\, f\, \hat{u}$$

Rearrange to get:

$$\hat{u}\,(e^{i\omega\Delta t} - 1) - \hat{v}\,\Delta t\, f = 0$$
$$\hat{u}\, f\,\Delta t + \hat{v}\,(e^{i\omega\Delta t} - 1) = 0$$

In matrix form, this is:

$$\begin{bmatrix} (e^{i\omega\Delta t} - 1) & -\Delta t\, f \\ \Delta t\, f & (e^{i\omega\Delta t} - 1) \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = 0$$

This set of eqs has a solution only if the determinant is zero. Then:

$$\left(e^{i\omega\Delta t}-1\right)^2 + \Delta t^2 \varsigma^2 = 0$$

Expand to get:

$$\underbrace{\left(e^{i\omega\Delta t}\right)^2}_{ax^2} \underbrace{- 2\, e^{i\omega\Delta t}}_{+\,bx} + \underbrace{1 + (\Delta t)^2 \varsigma^2}_{+\,c} = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This looks complicated but is actually just a simple quadratic eq. in $e^{i\omega\Delta t}$. Applying the standard quadratic formula yields:

$$e^{i\omega\Delta t} = \frac{2 \pm \sqrt{4 - 4\left(1 + \Delta t^2 \varsigma^2\right)}}{2}$$

$$= \frac{2 \pm \sqrt{4 \cdot \left[1 - \left(1 + \Delta t^2 \varsigma^2\right)\right]}}{2}$$

$$= 1 \pm i\, \Delta t\, \varsigma$$

Recall $e^{i\omega\Delta t} = \cos(\omega t) + i \sin(\omega t)$. Then we can equate real and imaginary parts to get:

$$\lambda \cos \omega_r \Delta t = 1$$
$$\lambda \sin \omega_r \Delta t = \pm \Delta t\, \varsigma$$

The factor λ is introduced to account for the possibility that the numerical solution may fictitiously grow or decay with time. (Formally we could have carried λ all the way through the derivation.)

- If $\lambda > 1$, the numerical solution continually grows with time. Such solutions are linearly unstable since $(u,v) \to \infty$ as $t \to \infty$.

- If $\lambda < 1$, the solution is damped, i.e. the system gradually slows down; $(u,v) \to 0$ as $t \to \infty$.

- If $\lambda = 1$, the original amplitude is preserved as $t \to \infty$. The solution is neutrally stable.

We can evaluate λ by squaring our two expressions and adding together:

$$\begin{array}{rl} \lambda^2 \cos^2(\omega_r \Delta t) &= 1 \\ + \;\lambda^2 \sin^2(\omega_r \Delta t) &= \Delta t^2 f^2 \\ \hline \lambda^2 \left[\sin^2(\omega_r \Delta t) + \cos^2(\omega_r \Delta t)\right] &= 1 + \Delta t^2 f^2 \end{array}$$

Since $\sin^2 + \cos^2 = 1$, we have

$$\lambda^2 = 1 + \Delta t^2 f^2$$

or

$$\boxed{\lambda = (1 + \Delta t^2 f^2)^{1/2}}$$

Then the solution is linearly unstable for nonzero f or Δt.

By revising the procedure slightly, we can create a neutrally stable scheme. We specify

$$u(t+\Delta t) = u(t) + f\,v(t)$$
$$v(t+\Delta t) = v(t) - f\,u(t+\Delta t)$$

Here we first update u, then use the updated value of u to compute the forcing for v. This is a simple type of implicit scheme since it uses information from the "future" time to compute the forcing.

Following the same type of analysis as we used for the explicit scheme, it can be shown that the implicit scheme is stable provided:

$$\Delta t^2 f^2 \le 4$$

For $f = 10^{-4}\ s^{-1}$, this yields $\Delta t \le 2 \cdot 10^4$ s (about 5½ hr).

<u>Advection schemes</u> (characterized by both <u>time</u> and <u>space</u>)

Forward-in-time, centered-in-space $\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$.

$u^{t+1} = u^{t} \mp u_i^{t} \Delta t \left[\frac{u_{i+1} - u}{2 \Delta x} \right.$

Centered-in-time, centered-in-space

2<u>nd</u> order in space: stable but short-wavelength features move too slowly; <u>computational</u> <u>dispersio</u>

4<u>th</u> order: much more accurate for short waves

Forward-in-time, upwind:

$u^{t+1} = u^{t} + u_{i\pm 1}^{t} \Delta t \left[\frac{u_{\cdot} - u_{\cdot}}{\Delta x} \right.$

linear gradients: stable but damped

higher-order greatly decreases damping & improves phase fidelity

Model Considerations

1) Coordinate systems — $\sigma = \left(\frac{z_{top} \; z}{z_{top} - z_{sfc}} \right)$

terrain good

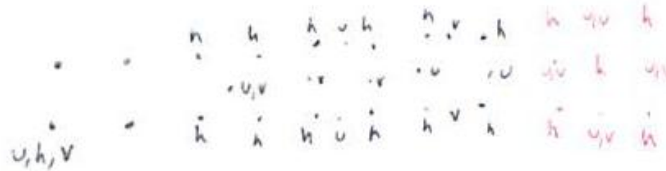
η

terrain in steps

— T, p
— w —
— T, p

2) Grid structures — staggered
unstaggered

A - B - C - D - E



C and d best

3) spatial + temporal discretization (centered schemes, one sided, forward-in-time

4. closures + parameterizations

5) Initial + Boundary Conditions